Elliptic Functions

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1 Motivation and Overview

In our previous lectures, we've analyzed the abelian extensions of $\mathbb{Q}$ in terms of the ideal group. Now we want to study extensions of imaginary quadratic fields $K = \mathbb{Q}(\sqrt{-D})$, and we will do this in terms of the arithmetic data of K. Since ideals of K are naturally lattices in $\mathbb{C}$, it is natural to study lattices and their own properties. In this lecture, we study lattices in $\mathbb{C}$, and study functions which are invariant under lattices. We study these elliptic functions explicitly, proving many great theorems about them. Then we give a few examples and show that we've generated all such elliptic functions. Then we give a description of elliptic curves over $\mathbb{C}$ as tori and briefly discuss the j-invariant.

2 Lattices and Elliptic Functions

Let L be a lattice in $\mathbb{C}$, that is, a subgroup of $\mathbb{C}$ that is a free abelian group of rank 2 and contains a basis for $\mathbb{C}$ as an $\mathbb{R}$ vector space. It is clear that all such lattices can be generated by two vectors w_1, w_2. We write $L = [w_1, w_2]$ and demand $Im(\frac{w_1}{w_2}) > 0$.

Definition 2.1. *We say a meromorphic function $f : \mathbb{C} \to \mathbb{C}$ is elliptic for a lattice L if $f(z + w) = f(z), \forall w \in L$.*

If f has no poles, then f is bounded, as it is well defined on $\mathbb{C}/L$, which is topologically a torus and therefore compact. Thus, f is bounded and entire, and therefore constant. So if we're interested in nonconstant elliptic functions, we must allow poles.

Theorem 2.2. *Let P be a fundamental parallelogram for $\mathbb{C}/L$ such that f has no poles on ∂P. (This can always be arranged) Then* $\displaystyle\sum_{residues\ in\ P} res(f) = 0$

Proof. By Cauchy's theorem (one of them):

$$2\pi i \sum Res(f) = \int_{\partial P} f(z)dz$$

However this integral is taken along the boundary of a parallelogram, and thus the contributions from opposite sides (which are traversed in opposite directions) cancel. $\square$

Corollary 2.3. *Nonconstant elliptic functions have at least two poles in an appropriately shifted parallelogram P whose boundary avoids the poles.*

We present another theorem along the same lines as the first.

Theorem 2.4. *Let $\{a_i\}$ be the set of points where f has a pole or zero in P. Suppose f has order m_i at a_i. Then $\sum m_i = 0$.*

Proof. $2\pi i \sum m_i = \int_{\partial P} \frac{f'(z)}{f(z)} dz$, and we can conclude as before. □

We present again another proof along a similar vein.

Theorem 2.5. *With the notation as before, $\sum m_i a_i \equiv 0 \pmod{L}$*

Proof. We look around for a function to integrate and after some time consider the following:

$$2\pi i \sum m_i a_i = \int_{\partial P} z \frac{f'(z)}{f(z)} dz$$

Now we break up the integral and use symmetry to conclude that it lies in $2\pi i L$ and conclude. □

With these theorems in place we prove the existence of nonconstant elliptic functions and explore their properties.

3 Constructing Elliptic Functions

We think very hard and then write down the following function:

$$\wp(z) = \frac{1}{z^2} + \sum_{w \in L}' \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right)$$

We really should show that this converges uniformly on compact sets not including lattice points, but don't. We hope the reader has a strong enough resolve to verify this for themselves. It isn't clear that $\wp$ is elliptic, but there are a few things which are clear. Clearly $\wp$ has double poles on L and nowhere else, and is even. Now consider the derivative:

$$\wp`(z) = -2 \sum_{w \in L}` \frac{1}{(w-z)^3}$$

Clearly $\wp`$ is elliptic and odd. The periodicity of $\wp`$ implies that the function $\wp(z+w_1) - \wp(z)$ is constant. We wish to show this constant is 0. Let $z = \frac{w_1}{2}$ and we obtain:

$$\wp(\frac{w_1}{2}) = \wp(\frac{-w_1}{2}) + C$$

However, by the evenness of $\wp$ we see that $C = 0$. We do the same song and dance for w_2 and obtain that $\wp$ is elliptic, which isn't obvious from its definition. It might be a little presumptuous to assume that we've generated all elliptic functions, but it turns out the rigidity of complex analysis forces this to be true. We refer the reader to Lang's elliptic functions for a proof.

Theorem 3.1. *The field of elliptic functions for L is generated by $\wp$ and $\wp'$.*

Now we search for a relation among $\wp$ and $\wp$‘ by brute force expansion about 0. Again, for details, see Lang. We define the function $\phi(z)$ as follows:

$$\phi(z) = \wp'(z)^2 - 4\wp(z)^3 + g_2\wp(z) + g_3$$

It turns out that this function has no poles or constant term (by inspecting the expansions) and is visibly elliptic, and therefore must be constant. To summarize the above, we have the following map.

$$(\wp, \wp') : \mathbb{C}/L \to E : y^2 = 4x^3 - g_2x - g_3$$

It can be shown that the right hand side has nonzero discriminant and therefore defines an elliptic curve. One natural question to ask is if this map is surjective. It certainly is, as $\wp(z) - \alpha$ has a double pole at 0 and therefore must have two roots corresponding to the two square roots of the cubic. In more advanced terminology, we see that $\wp$‘ seperates the two points z with $\wp(z) = \alpha$. Putting this together we say that $(\wp, \wp')$ defines a complex analytic isomorphism between $\mathbb{C}/L$ and $E(\mathbb{C})$.

4 Understanding the Group Law

One of the perks of the above construction is that it allows us to understand the somewhat mysterious group law on elliptic curves in terms of the much more well understood group law on $\mathbb{C}/L$. In particular, we will see that when we transport the group law on $\mathbb{C}/L$ to $E(\mathbb{C})$ it is precisely the one described by the secant line method that is usually first taught in the theory of elliptic curves.

Consider $u, v \in \mathbb{C}/L$. We want to understand $\wp(u + v)$ in terms of $\wp(u)$ and $\wp(v)$. Let $y = ax + b$ be the line containing the points $(\wp(u), \wp'(u))$ and $(\wp(v), \wp'(v))$. Then $\wp'(z) - (a\wp(z) + b)$ has a pole of order 3 at 0, which means it has 3 zeroes. Assuming for simplicity that they're distinct. By Theorem 2.5, we have that the third zero, w, must satisfy $u + v + w = 0$. Thus we have the following equality:

$$4x^3 - g_2x - g_3 - (ax + b)^2 = 4(x - \wp(u))(x - \wp(v))(x - \wp(w))$$

Comparing the coefficients of x^2 gives $\wp(u) + \wp(v) + \wp(w) = \frac{a^2}{4}$. We also have $a(\wp(u) - \wp(v)) = \wp'(u) - \wp'(v)$. These facts together give:

$$\wp(u + v) = -\wp(u) - \wp(v) + \frac{1}{4}\left(\frac{\wp'(u) - \wp'(v)}{\wp(u) - \wp(v)}\right)^2$$

Careful inspection reveals that this is the same as the secant line method the reader is probably familiar with being introducted to when learning the basic theory of elliptic curves.